2^{nd} exercise quantum mechanics Institut für theoretische Physik, JKU Linz
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5. A quantum-mechanical particle in an electromagnetic field is described by the Hamiltonian

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)^2 + V(\mathbf{r}).$$

Show that the probability density $\rho(\mathbf{r},t) = \psi^*(\mathbf{r},t)\psi(\mathbf{r},t)$ obeys a continuity equation of the form

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \nabla\cdot\mathbf{J}(\mathbf{r},t) = 0$$

and calculate the probability current $\mathbf{J}(\mathbf{r},t)$. Express the current in terms of the "canonical momentum operator"

$$\mathbf{\Pi} := \left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right).$$

6. Given is a gaussian wave package,

$$\psi(x,t) = \int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}A\exp[i(kx - \frac{\hbar k^2}{2m}t)]\exp[-(k-k_0)^2d^2].$$

where the constant A is determined by the normalization condition. Solve the integral and calculate the probability density of the wave package as a function of time. Interpret your results. Find an adequate value for A.

7. A function of a quantum-mechanical operator is defined by its Taylor series, *i.e.* a function of the momentum operator would be defined by the relation

$$\hat{F}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) = \sum_{n=0}^{\infty} a_n\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^n. \qquad (1)$$

Show under which conditions for the coefficients a_n and the wave functions ψ defined on an interval $]-\infty,+\infty[$ this operator is hermitian. Calculate explicitly for $n = 0,\ldots,3$, then try to generalize your findings to arbitrary n.

Hint: $\hat{F}$ is hermitian if $(\psi_i, \hat{F}\psi_k) = (\hat{F}\psi_i, \psi_k)$. Use integration by parts to "move over" the differential operator.

8. A photon is scattered from an electron at rest. What is the wavelength of the scattered photon as a function of the scattering angle θ if the incident photon has

   (a) wavelength $\lambda = 5000$ Å
   (b) energy $E = 0.52$ MeV.

Calculate the velocity v' and the scattering angle ϕ of the electron for $\theta = \pi/2$ in both of the above cases.