13. Consider two commuting quantum mechanical operators A and B, $[A, B] = 0$.

(a) Show that for any eigenfunction φ of B, the function $A\varphi$ also is an eigenfunction of B to the same eigenvalue.

(b) Show that A and B have a common set of eigenfunctions. Which conclusions can you draw for non-degenerate eigenvalues? What happens if the eigenvalues are degenerate? *Hint:* Introduce a unitary matrix D that transforms A to diagonal form. Then write down the equation $AB = BA$ in the space where A is diagonal ...

(c) To have a playground to apply these theorems, use them to find the eigenvalues and eigenvectors of

$$H = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} :$$

i. Find the matrix representation of the parity operator P in $\mathbb{R}^3$
ii. Show that H commutes with P.
iii. What are the eigenvalues and eigenvectors of P? (You don't need to *calculate* them - just *find* them :)
iv. What is the unitary matrix S that transforms P to diagonal form, $P_{\text{diag}} = S^{-1}PS$?
v. Calculate $S^{-1}HS$!
vi. Now find the eigenvectors of H.

14. A particle moves in a potential of the form

$$V(x) = -\frac{\hbar^2}{m} D\delta(x), \quad D > 0.$$

(a) Justify why the wave function is continuous at $x = 0$.

(b) Show that the first derivative of the wave function has a jump at $x = 0$,

$$\psi'(0^-) = \psi'(0^+) + 2D\psi(0).$$

(c) Solve the stationary Schrödinger equation for $E < 0$. Show that there is only one bound state. Calculate the energy of this state and the corresponding normalized wave function.

15. A quantum mechanical particle moves in the direction of the positive x-axis and hits a potential barrier of the form

$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ V_0 & \text{for } 0 < x < d \\ 2V_0 & \text{for } x > d \end{cases}, \quad V_0 > 0.$$

(a) Calculate the transmission coefficient of the particle for $E > 2V_0$

(b) Express the transmission coefficient in dimensionless variables. Make a sketch of the transmission coefficient as a function of the (dimensionless) energy. *Hint:* The following might be useful definitions for dimensionless units:

$$\epsilon := \frac{E}{V_0}, \ \eta := \sqrt{\frac{2mV_0 d^2}{\hbar^2}} > 0$$

16. You are given the differential operator $A := x\frac{d}{dx}$. Calculate the adjoint $A^\dagger$ of A.