21. The Schrödinger equation for periodic potentials I

(a) Show that for a periodic potential with period a, $V(x+a) = V(x)$, the Hamiltonian commutes with the translation operator T_a, $[T_a, H] = 0$. The operator T_a translates the whole system by a distance a and is defined by

$$T_a \psi(x) = \psi(x+a).$$

What is the consequence of this commutator relation?

(b) Show that from part (a) it follows that the solutions $\psi(x)$ of the Schrödinger equation for a periodic potential must be of the form

$$\psi(x) = e^{ikx} u(x) \;\text{ with }\; u(x) = u(x+h) \tag{1}$$

and a real-valued parameter k. This relation is called the *Bloch theorem*.
Hint: Use the fact that the translation operator does not change the norm of the wavefunction. What do you know about the operator product $T_a T_{a'}$?

(c) Consider a periodic system where the potential is so weak that you can neglect it — *i.e.* you can assume $V(x) = 0$, but the wavefunctions are still constrained to have the form given by equation (1). Calculate the eigenenergies of the system in dependence of the parameter k. Make a sketch of $E(k)$ in the region $[-\frac{\pi}{a}, +\frac{\pi}{a}]$. Why is it sufficient to constrain k to this region?
Hint: As $u(x+a) = u(x)$, the function $u(x)$ can be expressed as a Fourier series,

$$u(x) = \sum_{n=-\infty}^{\infty} u_n e^{iG_n x}, \;\text{ with }\; G_n = \frac{2\pi}{a} n.$$

22. The Schrödinger equation for periodic potentials II

Now consider a one-dimensional quantum system in the periodic potential

$$V(x) = -V_0 \sum_{n=-\infty}^{+\infty} \delta(x - na), \;\; V_0 > 0.$$

Derive an equation for the energy eigenvalues of the problem, and show how the equation can be solved graphically. Are there solutions for any value of k? Try to make a sketch of the function $E(k)$.

Hint: Use the usual Ansatz for the wavefunction $\psi(x)$ in the interval $[0, a]$. Then use the following conditions to determine the unknowns:

- Contituity of $\psi(x)$ at $x = 0$.
- The jump in $\psi'(x)$ at $x = 0$ (see problem 14 (b)).
- The Bloch theorem (equation (1) above).

23. A basis of angular momentum eigenstates is given by $\{|\tau l m_z\rangle\}$ (l, m_z quantum numbers of L^2 and L_z; τ is a set of quantum numbers, which belong to a set of self-adjoint operators T commuting with all components of $\vec{L}$. τ, l, m form a complete set of quantum numbers).

   (a) In which of the states $|\tau l m_z\rangle$ with fixed l does the uncertainty of the components L_x and L_y have the smallest value? Calculate this value.

   (b) Are there states in the considered state space, where all components of the concerning angular momentum have a 'sharp value' (*i.e.* no uncertainity)?

24. Consider an arbitrary physical system, whose four-dimensional state space is spanned by the four eigenstates $|l, m_z\rangle$ of the operators L^2 and L_z ($l = 0, 1; -l < m_z < l$). The eigenvalues are $l(l+1)\hbar^2$ and $m_z\hbar$, respectively, so that

$$\begin{aligned} L_\pm |l, m_z\rangle &= \hbar\sqrt{l(l+1) - m_z(m_z \pm 1)}|l, m_z \pm 1\rangle \\ L_+|l, l\rangle &= L_-|l, -l\rangle = 0. \end{aligned} \tag{2}$$

   (a) Write the eigenstates $|l, m_x\rangle$ of L^2 and L_x in terms of states $|l, m_z\rangle$.

   (b) Consider a system in the normalized state (notation $|l, m_z\rangle$)

$$|\Psi\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle + \delta|0,0\rangle.$$

   What are the probabilities, that measurements of L^2 and $L_z$give $2\hbar^2$ and $\hbar$, respectively? Calculate also the average value of L_z, when the system is in the state $|\Psi\rangle$, and the probabilities of each possible value this observable can have.