29. (a) Show that the current density $\mathbf{j}(\mathbf{r})$ associated with a quantum mechanical particle can be written as

$$\mathbf{j}(\mathbf{r}) = \frac{\hbar}{m}\rho(\mathbf{r})\nabla\zeta(\mathbf{r}),$$

where $\rho(\mathbf{r})$ is the probability density and $\zeta(\mathbf{r})$ is the phase of the wave function.
Hint: You can write any complex function in the form $\psi(\mathbf{r}) = \alpha(\mathbf{r})e^{i\zeta(\mathbf{r})}$ with real functions α and ζ

(b) Calculate the current density for the stationary states of the hydrogen atom and discuss the result. Explicitly calculate $j(r)$ for the state $(n, l, m) = (2, 1, 1)$ and make a sketch.

(c) Now assume that there is a homogenous magnetic field along the z-axis. Calculate $\mathbf{j}(\mathbf{r})$ for this case. As an approximation, you can assume that the additional (small) $\mathbf{B}$-field does not change the wave function, and you can also neglect terms proportional to $\mathbf{A}^2$. Compare your results to the situation without the magnetic field.
Hint: You already calculated the general expression for the current density in problem no. 5.

30. In a simple model for the Deuterium nucleus, the interaction between the proton and the neutron is described by the potential

$$V(r) = -Ae^{-\frac{r}{a}}, \quad \text{with } A > 0,\ a > 0.$$

Reduce the Schrödinger equation of the problem to an effective one-body equation and solve it for $\ell = 0$, where ℓ is the quantum number associated with the angular momentum. Make a sketch of the wave function of the lowest energy state (you don't have to correctly normalize the wave function, and you don't need to calculate the energy).
Hint: Use the substitution $y = e^{-\frac{r}{2a}}$ - the resulting differential equation is a very prominent one.

31. A particle is moving in a spherical-symmetric potential $V(r)$. The wave function is given by

$$u(\vec{r}) = (x + y + 3z)f(r)$$

with $r = \sqrt{x^2 + y^2 + z^2}$.

(a) Is $u(\vec{r})$ an eigenstate of L^2? If it's true, what is the eigenvalue?

(b) A measurement of L_z is performed. Which values of L_z will appear and what are the probabilities for these values?

(c) Assuming that $u(\vec{r})$ is a solution of the Schrödinger equation with eigenvalue E. Express the potential depending on E and $f(r)$.

32. Considering the hydrogen atom without Spin and relativistic corrections. The state vector to a certain point of time t_0 is given by

$$|\psi\rangle = \frac{1}{6}\{4|100\rangle + 2|200\rangle - i|210\rangle + \sqrt{10}|21,-1\rangle + (1-2i)|32,-1\rangle\}$$

where $|nlm_l\rangle, n \in \mathbb{N}; l = n-1, n-2, .., 0; m_l = l, l-1, ..., -l$ energy eigenvalues of the hydrogen atom. Calculate for this point of time

(a) the probability finding the measured energy value E_1 *oder* E_2.

(b) the probability finding for the squared angular momentum the measured value b_0 *oder* b_2 $(b_l = l(l+1)\hbar^2)$

(c) the probability for the pair of measured values $\{E_2, b_1\}$ assuming

(i) that energy and angular momentum will be measured simultaneously

(ii) that the energy will be measured first and the angular momentum immediately afterwards