33. Consider a three-dimensional harmonic oscillator, which is given by the Hamiltonian

$$H = \sum_{i=1}^{3} H_i, \quad H_i = \frac{1}{2m}\left(P_i^2 + m^2\omega^2 X_i^2\right),$$

where X_i and P_i are the position and momentum operators in x, y, z-direction.

Express H in terms of the ladder operators $a_i, a_i^\dagger$ and derive an expression for the energy eigenvalues E_n in terms of the operators $N_i := a_i^\dagger a_i$. What is the degeneracy of the state E_n? How can one generate an arbitrary quantum state out of the vacuum state $|0\rangle$?

34. For the three-dimensional harmonic oscillator as defined in problem 33, express the angular momentum operator $\mathbf{L} = \mathbf{R} \times \mathbf{P}$ in terms of the ladder operators.

(a) Show that

$$L_z = \hbar(a_+^\dagger a_+ - a_-^\dagger a_-) \quad \text{and} \quad L_\pm = \pm\hbar\sqrt{2}(a_\mp a_z^\dagger - a_\pm^\dagger a_z)$$

with the definitions

$$L_\pm := (L_x \pm iL_y) \quad \text{and} \quad a_\pm^\dagger := \frac{1}{\sqrt{2}}(a_x^\dagger \pm i a_y^\dagger), \; a_\pm := \frac{1}{\sqrt{2}}(a_x \mp i a_y).$$

(b) Express the Hamiltonian H in terms of the ladder operators $a_\pm^\dagger, a_\pm, a_z^\dagger$ and a_z.

(c) How can you generate an arbitrary state out of the vacuum state $|0\rangle$ using $a_\pm^\dagger$ and $a_z^\dagger$? Is this state also an eigenstate of L_z? If yes, what is the corresponding eigenvalue?

35. For the one-dimensional harmonic oscillator, calculate the matrix representation of the operators x, x^2, p and p^2 in the basis of its eigenfunctions. Use your results to calculate $(\Delta x)^2(\Delta p)^2$ for the three lowest states.

36. (a) Formulate the Schrödinger equation of a linear (1-dim.) harmonic oscillator in momentum representation. Describe the appropriate eigenvalues E_n and eigenfunction $\tilde{u}_n \equiv \langle p|n\rangle$.

(b) A state vector of this harmonic oscillator is given to a certain point of time by a linear combination of his energy eigenstates $|0\rangle, |1\rangle$

$$|\psi\rangle = \sqrt{\frac{2}{3}}|0\rangle - i\frac{1}{\sqrt{3}}|1\rangle.$$

Calculate for this point of time uncertainty relation $\Delta x \cdot \Delta p$.