Low-temperature Thermodynamics of the Unitary Fermi Gas

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- Collective and single-particle excitations
- Low-temperature thermodynamics
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Unitary Fermi gas of atoms

Let us consider a gas of atomic fermions with two equally-populated spin components: $n_\uparrow = n_\downarrow$.

The system is <u>dilute</u> if the effective radius r_e of the inter-atomic potential is much smaller than the average interparticle separation $d = n^{-1/3}$, with $n = n_\uparrow + n_\downarrow$, namely

$$r_e \ll d \; . \tag{1}$$

The system is <u>strongly-interacting</u> if the scattering length a of the inter-atomic potential greatly exceeds the average interparticle separation $d = n^{-1/3}$, i.e.

$$d \ll |a| \; . \tag{2}$$

The unitarity regime* is characterized by both these conditions:

$$r_e \ll d \ll |a| \; . \tag{3}$$

Under these conditions the dilute but strongly-interacting Fermi gas is called <u>unitary Fermi gas</u>.

*S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **80**, 1215 (2008).

Ideally, the unitarity limit corresponds to

$$r_e = 0 \qquad \text{and} \qquad a = \pm\infty \, . \tag{4}$$

The only length characterizing the Fermi gas in the unitarity limit is the average interparticle distance $d = n^{-1/3}$.

In this case the ground-state energy per particle must be

$$\frac{E_0}{N} = \xi \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n^{2/3} = \xi \; \frac{3}{5} \; \epsilon_F \; , \tag{5}$$

with ϵ_F Fermi energy of the ideal gas and ξ a universal unknown parameter.

Monte Carlo calculations and experimental data with dilute and ultracold atoms suggest* that the unitary Fermi gas is a superfluid with $\xi \simeq 0.4$.

*S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **80**, 1215 (2008).

Collective and single-particle excitations

We model* the many-body quantum Hamiltonian $\hat{H}$ of the uniform unitary Fermi gas with the simple effective Hamiltonian

$$\hat{H} = E_0 + \sum_{\mathbf{q}} \epsilon_{col}(q) \ \hat{b}^+_{\mathbf{q}} \hat{b}_{\mathbf{q}} + \sum_{\sigma=\uparrow,\downarrow} \sum_{\mathbf{p}} \epsilon_{sp}(p) \ \hat{c}^+_{\mathbf{p}\sigma} \hat{c}_{\mathbf{p}\sigma} \ , \tag{6}$$

where

$$E_0 = \frac{3}{5} \xi N \epsilon_F \tag{7}$$

is the ground-state energy,

$$\epsilon_{col}(q) \tag{8}$$

is the energy of the bosonic collective excitations, and

$$\epsilon_{sp}(p) \tag{9}$$

is the energy of the fermionic single-particle excitations.

*Inspired by the Landau theory of elementary excitations: L.D. Landau and E.M. Lifshits, *Statistical Physics*, Part 2, vol. 9 (Butterworth-Heinemann, London, 1980).

Recently we have found* the dispersion relation of collective elementary excitations as

$$\epsilon_{col}(q) = \sqrt{c_1^2 q^2 + \frac{\lambda}{4m^2} q^4} \ , \qquad (10)$$

where

$$c_1 = \sqrt{\frac{\xi}{3}} \ v_F \ , \qquad (11)$$

is the zero-temperature first sound velocity, with $v_F = (\hbar/m)(3\pi^2 n)^{1/3}$ the Fermi velocity of a noninteracting Fermi gas. The term with λ takes into account the increase of kinetic energy in the Fermi system due the spatial variation of the density. We use $\lambda = 0.25$.

Expanding the dispersion relation (11) for low momenta we get

$$\epsilon_{col}(q) = c_1 q + \frac{\lambda}{8m^2 c_1} q^3 \ , \qquad (12)$$

where the linear term is the familiar phonon dispersion relation (the so-called Bogoliubov-Anderson mode) while the cubic correction depends on both the sound velocity c_1 and the parameter λ.

*L.S. and F. Toigo, Phys. Rev. A **78**, 053626 (2008); L.S., F. Ancilotto, and F. Toigo, Laser Phys. Lett. **7**, N. 1, 78 (2010).

At zero temperature the fermionic single-particle excitations can be written as

$$\epsilon_{sp}(p) = \sqrt{(\frac{p^2}{2m} - \zeta\epsilon_F)^2 + \Delta_0^2} \tag{13}$$

where ζ is a parameter which takes into account the interaction between fermions with ϵ_F the Fermi energy of the ideal Fermi gas.

According to recent Monte Carlo results*: $\zeta \simeq 0.9$ and $\gamma = \Delta_0/\epsilon_F \simeq 0.45$.

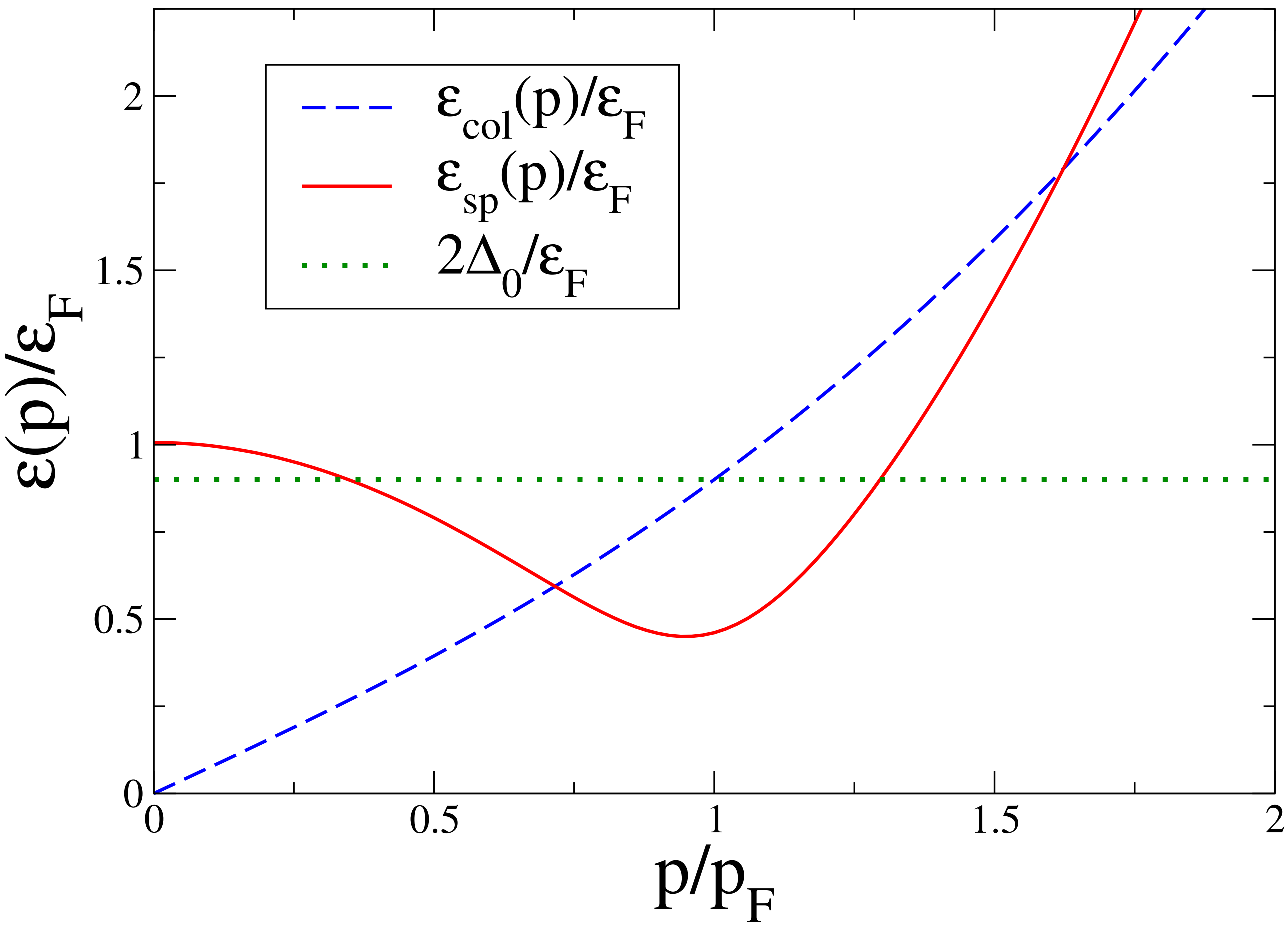
Expanding this dispersion relation around the minimum momentum $p_0 = \sqrt{2m\mu} = \zeta^{1/2} p_F$, with $p_F = \sqrt{2m\epsilon_F}$ the Fermi momentum of the ideal Fermi gas, we find

$$\epsilon_{sp}(p) = \Delta_0 + \frac{1}{2m_0}(p - p_0)^2 \, , \tag{14}$$

where the effective mass m_0 is given by

$$m_0 = \frac{m\Delta_0}{2\zeta\epsilon_F} \, . \tag{15}$$

*P. Magierski, G. Wlazlowski, A. Bulgac, and J.E. Drut, Phys. Rev. Lett. **103**, 210403 (2009).



Elementary excitations of the unitary Fermi gas: bosonic collective excitations $\epsilon_{col}(p)$ (dashed line) and fermionic single-particle excitations $\epsilon_{sp}(p)$ (solid line). The collective mode $\epsilon_{col}(p)$ decays in the single-particle continuum when there is the breaking of Cooper pairs, namely for $\epsilon_{col}(p) = 2\Delta_0$ (dotted line).

Low-temperature thermodynamics

At very low temperature the thermodynamic properties of the superfluid unitary Fermi gas can be obtained from the collective spectrum and considering an ideal Bose gas of elementary excitations with the bosonic distribution

$$f_B(q) = \langle \hat{b}^+_{\mathbf{q}} \hat{b}_{\mathbf{q}} \rangle = \frac{1}{e^{\epsilon_{col}(q)/k_B T} - 1} , \tag{16}$$

where $\langle \hat{A} \rangle = Tr[\hat{A} e^{-\hat{H}/k_B T}]/Tr[e^{-\hat{H}/k_B T}]$ is the thermal average of the operator $\hat{A}$ with T the absolute temperature and k_B is the Boltzmann constant.

As T increases also the fermionic single-particle excitations become important. Thus there is also the effect of an ideal Fermi gas of single-particle excitations with the fermionic distribution

$$f_F(p) = \langle \hat{c}^+_{\mathbf{p}\sigma} \hat{c}_{\mathbf{p}\sigma} \rangle = \frac{1}{e^{\epsilon_{sp}(p)/k_B T} + 1} , \tag{17}$$

which is spin independent.

The Helmholtz free energy F of any thermodynamic system is given by

$$F = -k_B T \ln \mathcal{Z} \ , \tag{18}$$

where

$$\mathcal{Z} = Tr[e^{-\hat{H}/k_B T}] \ , \tag{19}$$

is the partition function of the system.

Using our effective Hamiltonian the free energy of our unitary Fermi gas can be written as $F = F_0 + F_{col} + F_{sp}$, where

$$F_0 = \frac{3}{5}\xi N \epsilon_F \ , \tag{20}$$

$$F_{col} = k_B T \sum_{\mathbf{q}} \ln \left[1 - e^{-\epsilon_{col}(q)/(k_B T)} \right] \ , \tag{21}$$

$$F_{sp} = -2 \ k_B T \sum_{\mathbf{p}} \ln \left[1 + e^{-\epsilon_{sp}(p)/(k_B T)} \right] \ , \tag{22}$$

where the factor 2 is due to the two spin components.

The total Helmholtz free energy of the low-temperature unitary Fermi gas can be then written as

$$F = N\epsilon_F \Phi\left(\frac{T}{T_F}\right) \ , \tag{23}$$

where $\Phi(x)$ is a function of the scaled temperature $x = T/T_F$, with $T_F = \epsilon_F/k_B$, given by

$$\begin{aligned} \Phi(x) &= \frac{3}{5}\xi + \frac{3}{2}x\int_0^{\eta_{cut}} \ln\left[1 - e^{-\tilde{\epsilon}_{col}(\eta)/x}\right]\eta^2 d\eta \\ &- 3x\int_0^{+\infty} \ln\left[1 + e^{-\tilde{\epsilon}_{sp}(\eta)/x}\right]\eta^2 d\eta \ . \end{aligned} \tag{24}$$

Here the discrete summations have been replaced by integrals, $\tilde{\epsilon}_{col}(\eta) = \sqrt{\eta^2(\lambda\eta^2 + 4\xi/3)}$, and $\tilde{\epsilon}_{sp}(\eta) = \sqrt{(\eta^2 - \zeta)^2 + \gamma^2}$.

In the first integral which appears in $\Phi(x)$ there is an upper bound η_{cut}. Collective mode decays in the single-particle continuum when there is the breaking of Cooper pairs, namely for $\epsilon_{col}(q) = 2\Delta_0$, thus in scaled units when η_{cut} satisfies the equation

$${\eta_{cut}}^2(\lambda {\eta_{cut}}^2 + \frac{4}{3}\xi) = 4\gamma^2 \ . \tag{25}$$

By using the expansions for the elementary excitations, adopting the Maxwell-Boltzmann distribution for fermionic single-particles instead of the Fermi-Dirac one, and setting $\eta_{cut} = +\infty$ one gets the approximate formula

$$\Phi(x) \simeq \frac{3}{5}\xi - \frac{\pi^4\sqrt{3}}{80\ \xi^{3/2}}x^4 + \frac{\lambda\pi^6 3\sqrt{3}}{896\ \xi^{7/2}}x^6 - \frac{3\sqrt{2\pi}}{2}\zeta^{1/2}\gamma^{1/2}x^{3/2}e^{-\gamma/x}\ . \qquad (26)$$

Under the further assumption that $\lambda = 0$ this formula becomes exactly the simple model* proposed Bulgac, Drut, and Magierski (BDM)

$$\Phi(x) \simeq \frac{3}{5}\xi - \frac{\pi^4\sqrt{3}}{80\ \xi^{3/2}}x^4 - \frac{3\sqrt{2\pi}}{2}\zeta^{1/2}\gamma^{1/2}x^{3/2}e^{-\gamma/x}\ . \qquad (27)$$

*A. Bulgac, J.E. Drut, and P. Magierski, Phys. Rev. Lett **96**, 090404 (2006).

From the Helmholtz free energy F we can immediately obtain the chemical potential

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = \epsilon_F [\frac{5}{3}\Phi\left(\frac{T}{T_F}\right) - \frac{2}{3}\frac{T}{T_F}\Phi'\left(\frac{T}{T_F}\right)] \, , \tag{28}$$

where $\Phi'(x) = \frac{d\Phi(x)}{dx}$ and one recovers $\mu_0 = \xi\epsilon_F$ in the limit of zero-temperature

The entropy S is related to the free energy F by the formula

$$S = -\left(\frac{\partial F}{\partial T}\right)_{N,V} = -Nk_B\Phi'\left(\frac{T}{T_F}\right) \; . \tag{29}$$

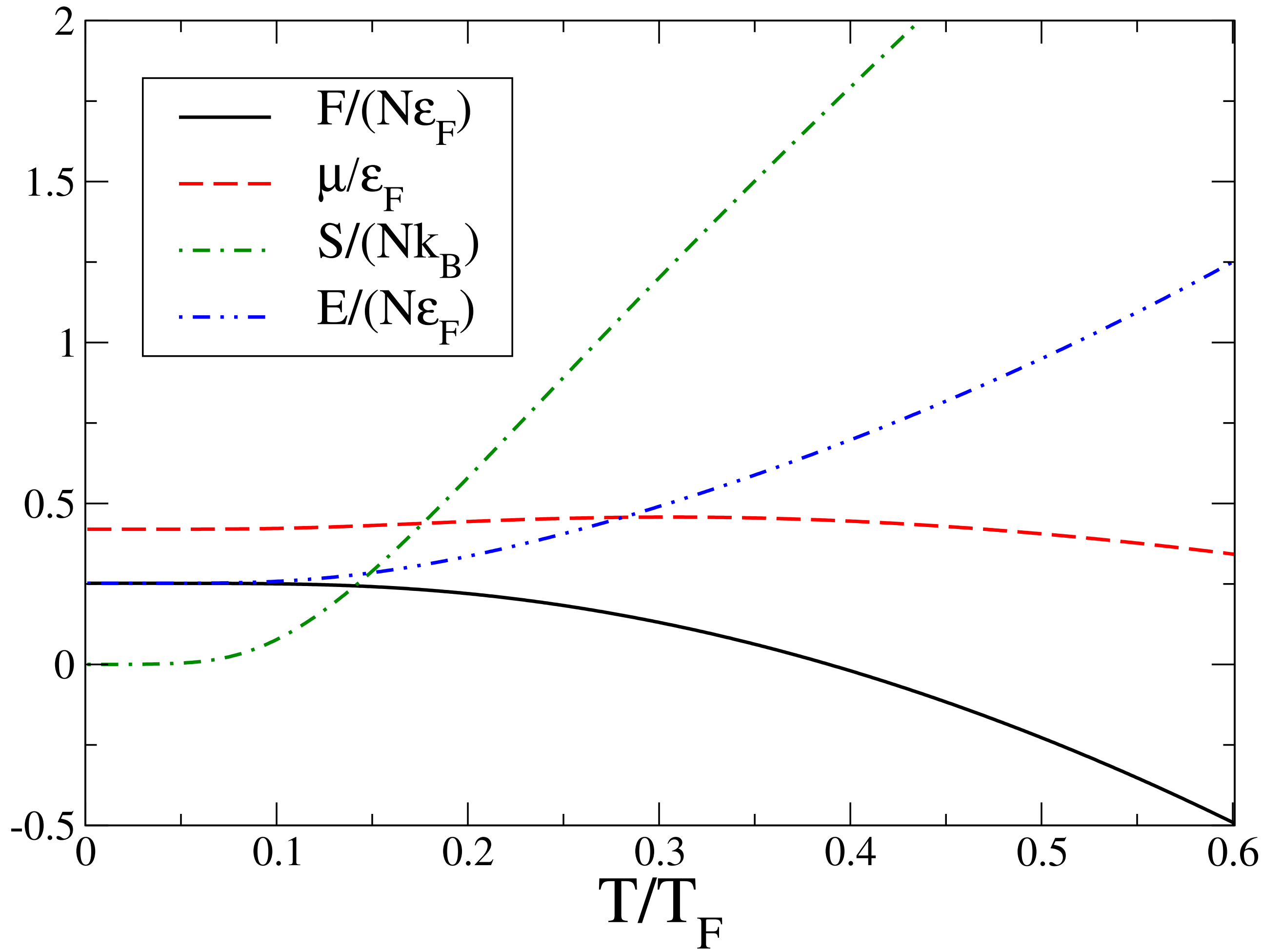
In addition, the internal energy E, given by

$$E = F + TS = N\epsilon_F\left[\Phi\left(\frac{T}{T_F}\right) - \frac{T}{T_F}\Phi'\left(\frac{T}{T_F}\right)\right] \; . \tag{30}$$

Important: our model does not predict a phase-transition. It is a low-temperature model for the superfluid phase. A phase transition for this uniform system has been predicted* and observed† at $T_c \simeq 0.15\ T_F$.

*E. Burovski, N. Prokof'ev, B. Svistunov, and M. Troyer, PRL **96**, 160402 (2006).

†S. Nascimbene *et al.*, Nature **463**, 1057 (2010).



Thermodynamical quantities of the unitary Fermi gas deduced from our model. Zero-temperature parameters of elementary excitations: $\xi = 0.42$, $\lambda = 0.25$, $\zeta = 0.9$, and $\gamma = 0.45$ (from which $\eta_{cut} = 1$).

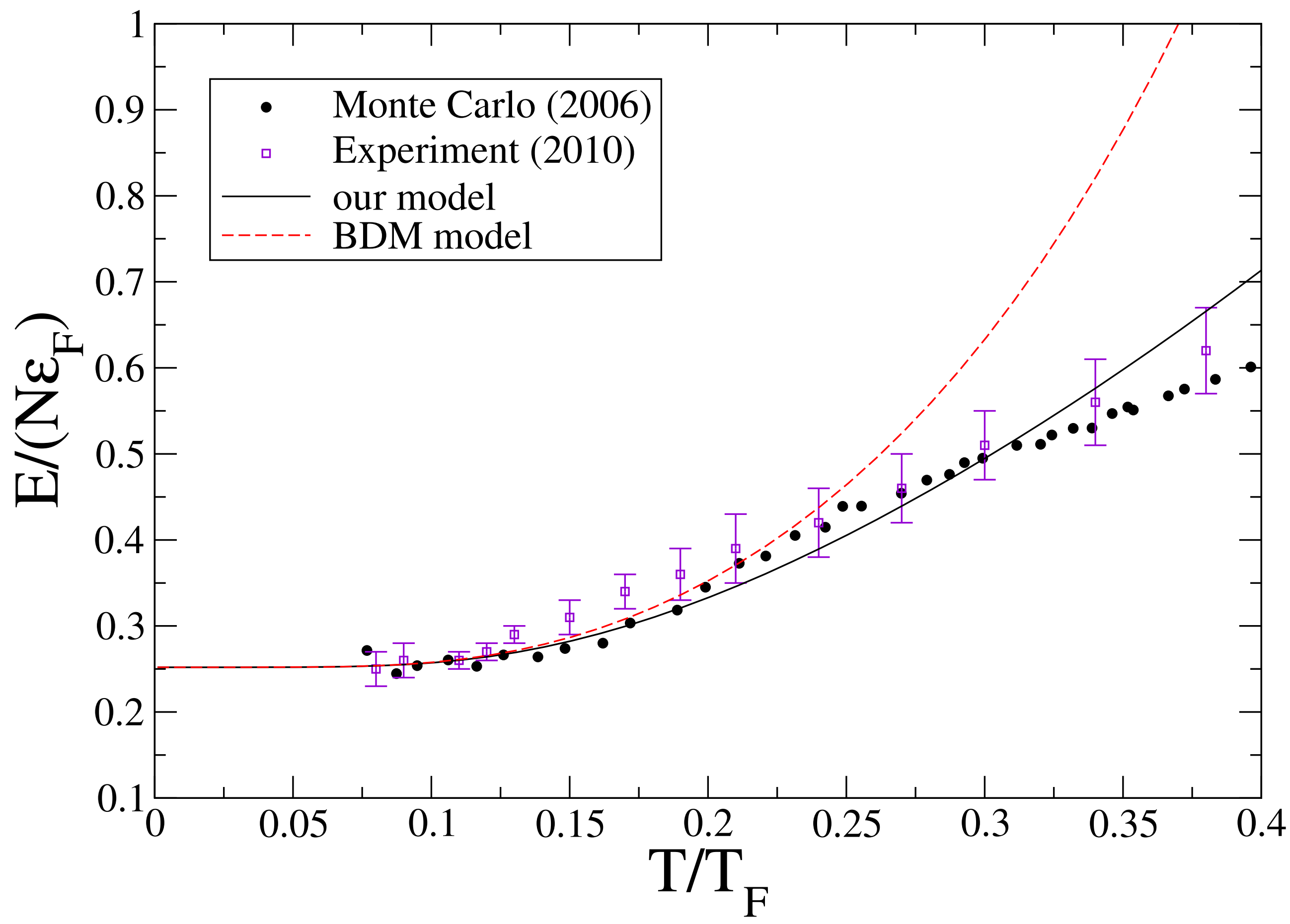
Comparison with MC results and experimental data

It is interesting to compare our model with other theoretical approaches and also with the available experimental data.

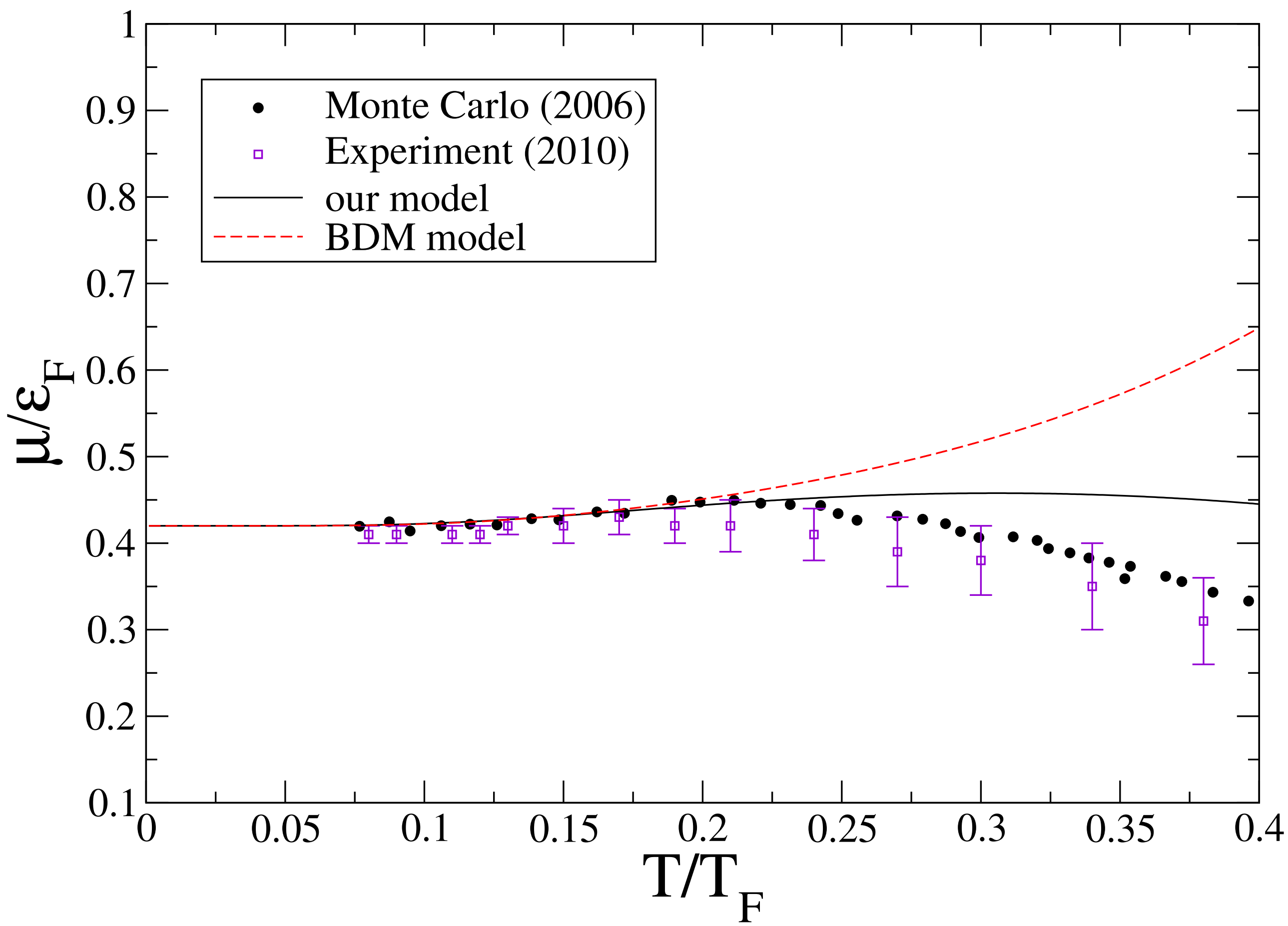
In the next two figures we report the data of internal energy E and chemical potential μ obtained by Bulgac, Drut, and Magierski[*] with their Monte Carlo simulations of the atomic unitary gas.

We insert also the very recent experimental data of Horikoshi *et al.*[†] for the unitary Fermi gas of ^{6}Li atoms but extracted from the gas under harmonic confinement.

[*] A. Bulgac, J.E. Drut, and P. Magierski, Phys. Rev. Lett **96**, 090404 (2006).
[†] M. Horikoshi, S. Nakajima, M. Ueda, and T. Mukaiyama, Science **442**, 327 (2010).



Atomic unitary Fermi gas. Scaled internal energy $E/(N\epsilon_F)$ as a function of the scaled temperature T/T_F. Filled circles: Monte Carlo simulations of BDM. Open squares with error bars: experimental data of Horikoshi *et al.* Solid line: our model. Dashed line: BDM model.



Atomic unitary Fermi gas. Scaled chemical potential $E/(N\epsilon_F)$ as a function of the scaled temperature T/T_F. Filled circles: Monte Carlo simulations of BDM. Open squares with error bars: experimental data of Horikoshi *et al.* Solid line: our model. Dashed line: BDM model.

Dilute neutron matter: MC vs our thermo model

The dilute neutron matter is predicted to fill the crust of neutron stars*

MC data have been produced for dilute neutron matter at finite temperature by Wlazlowski and Magierski† with the density $n = 0.003$ fm^{-3}, i.e. inter-particle separation $d = n^{-1/3} = 6.93$ fm.

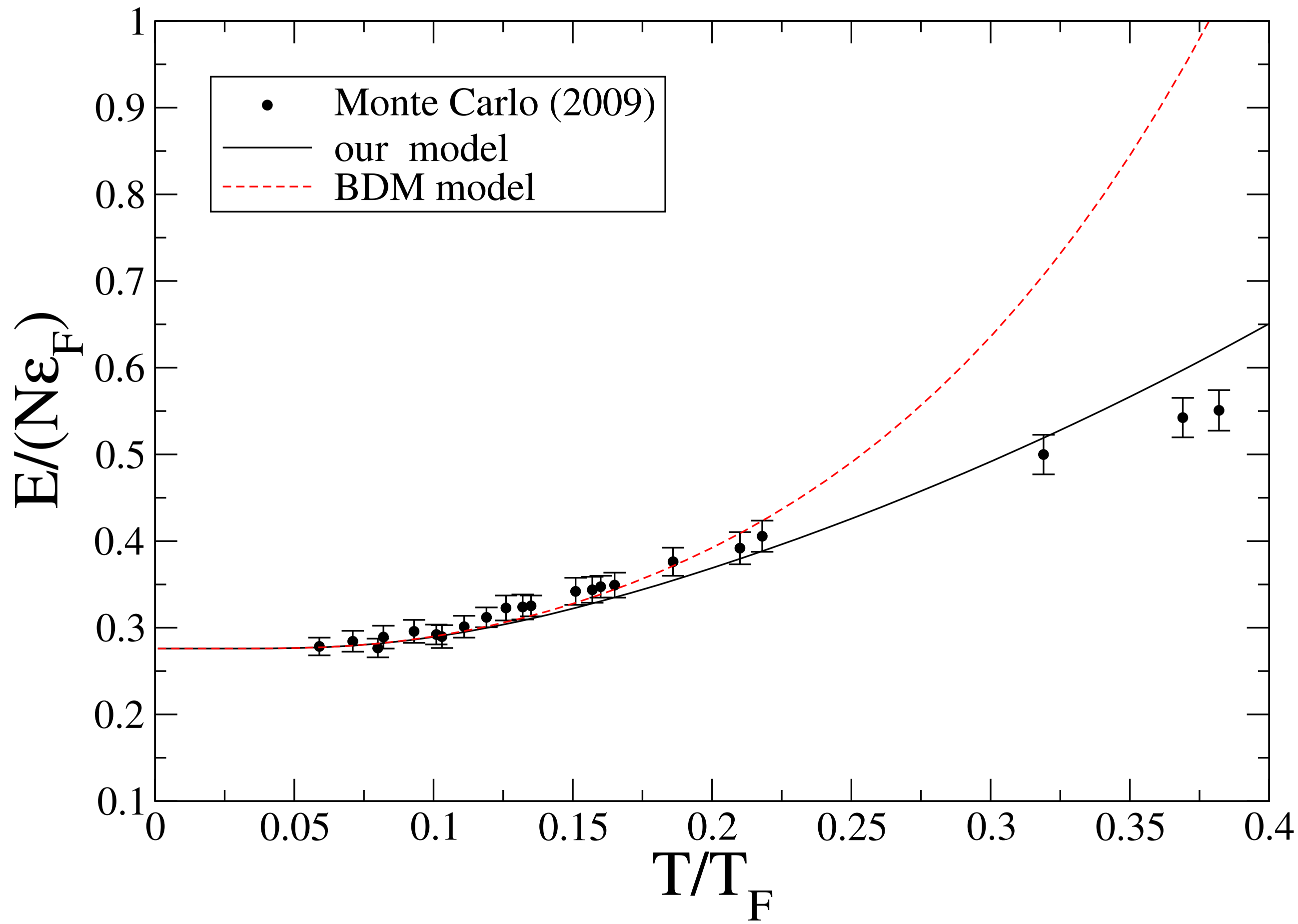
For the neutron-neutron interaction: $r_e \simeq 2.8$ fm and $a \simeq -18.5$ fm. This means that in the calculations: $r_e < d < |a|$.

Thus this dilute neutron matter is close but not equal to the unitarity Fermi gas ($r_e \ll d \ll |a|$). The zero-temperature parameters of elementary excitations are slightly different from those of the unitary Fermi gas.

Fermi temperatures. Neutron matter: $T_F \simeq 5 \cdot 10^{10}$ Kelvin; ultracold atomic vapors: $T_F \simeq 10^{-7}$ Kelvin.

*A. Schwenk and C.J. Pethick, PRL **95**, 160401 (2005).

†G. Wlazlowski and P. Magierski, Int. J. Mod. Phys. E **18**, 919 (2009).



Dilute neutron matter at the density $n = 0.003$ fm^{-3}. Scaled internal energy $E/(N\epsilon_F)$ as a function of the scaled temperature T/T_F. Filled circles: Monte Carlo simulations. Solid line: our model. Dashed line: BDM model. Zero-temperature parameters of elementary excitations (extracted from MC spectral weight function): $\xi = 0.46$, $\lambda = 0.25$, $\zeta = 0.82$, and $\gamma = 0.29$ (from which $\eta_{cut} = 0.68$).

Conclusions

- In our model the low-temperature thermodynamics is obtained by using the zero-temperature elementary excitations.

- Our model does not predict the superfluid-normal phase transition, but it works quite well in the superfluid regime (also slightly above the expected critical temperature $T_c \simeq 0.15$).

- The improvement of our model with respect to the BDM one is mainly due to the inclusion of the ultraviolet cut-off η_{cut} in the bosonic spectrum, related to the breaking of Cooper pairs.