Gallager error correcting codes for binary asymmetric channels

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Using a statistical mechanics approach we study error correction through Gallager codes on binary asymmetric channels. For these channels the probability p that a bit with value $+1$ flips is different than the probability q that a bit with value 0 flips. In the case $p = q$ we retrieve the binary symmetric channel and in the case $p = 0$ we get the Z-channel. The probability distribution of noise variables is identified with a spin system on a finitely connected hypergraph with infinitely strong interactions and random external fields. Because of the asymmetry of the channel, we get in the a-posteriori probability distribution of noise variables an explicit dependence on the retrieved word. The random fields are determined by the log-likelihood ratios of the individual noise variables. We scale the log-likelihood ratios of the noise bits to find a finite temperature phase diagram. Using replica symmetry we find the mean field solution of the corresponding spin system. Because of the explicit dependence on the retrieved message, we get a selfconsistent distributional equation containing two types of cavity fields. We determine the finite temperature phase diagram. This phase diagram is characterized by a ferromagnetic, paramagnetic and spin glass state. At low noise levels the ferromagnetic state is the only stable state, reflecting the success of decoding schemes using belief propagation. Calculation of the entropy leads to the threshold for Gallager encoding on asymmetric channels using any decoding scheme. At low temperatures we find that the message passing algorithms stop converging. This non-convergence can be linked with the failure of the endogeny property introduced in probability theory of recursive distributional equations. Assuming one-step replica symmetry, we can link the non-convergence of the message passing algorithm with the appearance of a solution with a positive complexity.