## Applications of Fractional Calculus to Equilibrium and Nonequilibrium Statistical Mechanics

R. Hilfer*,**

* ICP, Universität Stuttgart, 70569 Stuttgart, Germany
** Institut für Physik, Universität Mainz, 55099 Mainz, Germany

Applications of fractional calculus to anomalous transport, non-Fickian diffusion, nonexponential relaxation, critical phenomena of continuous phase transitions or non-Gaussian random walks have found increasing scientific attention in recent years [1, 2]. Such applications raise a crucial question: How does the noninteger order $\alpha$ of the fractional derivatives or integrals arise from fundamental theory, and can it be measured in an independent experiment? Many works, particularly in the engineering literature, ignore this question and treat the fractional order as a phenomenological parameter that cannot be measured in an independent experiment. The first concrete example of an answer to the question above was obtained within the classification theory of continuous phase transitions where the fractional order of derivatives was shown to be related to the asymptotic properties of equilibrium measures at critical points [3–8]. The second example concerns fractional time derivatives, and it answers the related question: How can fractional time derivatives arise in equations of motion, when it is an elementary fact that all the fundamental laws of physics are time translation invariant, and the infinitesimal generator of time translations are first order time derivatives ? This question was addressed and investigated extensively by several different methods in [6, 9–13]. Recently, these results and methods have been used to generalize a theorem by Ruelle within the thermodynamic formalism of Markov shifts [14]. A concrete and mathematically rigorous example for the identification of the fractional order $\alpha$ within a microscopic model was found for continuous time random walks with Mittag-Leffler waiting time density in [15] and it has been discussed extensively in [16–18]. Continuous time random walks with Mittag-Leffler waiting time density correspond rigorously to a diffusion equation, and more generally to a generalized master equation, with fractional time derivative.

References available at `www.icp.uni-stuttgart.de/~hilfer/publikationen`

# References

[1] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific Publ. Co., Singapore, 2000.

[2] R. Klages et al., *Anomalous Transport*, Wiley-VCH, Weinheim, 2008.

[3] R. Hilfer, *Thermodynamic scaling derived via analytic continuation from the classification of Ehrenfest*, Physica Scripta **44** (1991), 321.

[4] ———, *Multiscaling and the classification of continuous phase transitions*, Phys. Rev. Lett. **68** (1992), 190.

[5] ———, *Scaling theory and the classification of phase transitions*, Mod. Phys. Lett. B **6** (1992), 773.

[6] ———, *Classification theory for anequilibrium phase transitions*, Phys. Rev. E **48** (1993), 2466.

[7] ———, *Absence of hyperscaling violations for phase transitions with positive specific heat exponent*, Z. Physik B **96** (1994), 63.

[8] ———, *Fractional calculus and regular variation in thermodynamics*, Applications of fractional calculus in physics, 2000, pp. 429.

[9] ———, *Fractional dynamics, irreversibility and ergodicity breaking*, Chaos, Solitons & Fractals **5** (1995), 1475.

[10] ———, *Foundations of fractional dynamics*, Fractals **3** (1995), 549.

[11] ———, *An extension of the dynamical foundation for the statistical equilibrium concept*, Physica A **221** (1995), 89.

[12] ———, *Fractional time evolution*, Applications of fractional calculus in physics, 2000, pp. 87.

[13] ———, *Remarks on fractional time*, Time, quantum and information, 2003, pp. 235.

[14] D. Ruelle, *Thermodynamic formalism*, Cambridge University Press, Cambridge, 2004.

[15] R. Hilfer and L. Anton, *Fractional master equations and fractal time random walks*, Phys.Rev.E, Rapid Commun. **51** (1995), R848.

[16] R. Hilfer, *Exact solutions for a class of fractal time random walks*, Fractals **3(1)** (1995), 211.

[17] ———, *On fractional diffusion and its relation with continuous time random walks*, Anomalous diffusion: From basis to applications, 1999, pp. 77.

[18] ———, *On fractional diffusion and continuous time random walks*, Physica A **329** (2003), 35.